Midterm exam Linear Algebra II

Thursday 09/03/2023, 18:30–20:30

1 $(8 = 3 + 3 + 2$ pts) **Span, linear independence and basis**

Let $\mathcal{V}$ be a vector space over the field $\mathbb{F}$. Suppose that $\{u_1, u_2, \ldots, u_r\}$ is a basis of $\mathcal{V}$. Define the new vector $v = c_1u_1 + c_2u_2 + \cdots + c_ru_r \in \mathcal{V}$, where $c_1, c_2, \ldots, c_r \in \mathbb{F}$ are scalars with $c_1 \neq 0$.

(a) Show that $\operatorname{span}(v, u_2, \ldots, u_r) = \mathcal{V}$.

(b) Show that the vectors $v, u_2, \ldots, u_r$ are linearly independent.

(c) Is $\{v, u_2, \ldots, u_r\}$ a basis of $\mathcal{V}$?

2 $(10 = 2 + 3 + 3 + 2$ pts) **Linear transformations**

Consider the $\mathbb{R}$-vector space $\mathbb{R}^{n\times n}$ of real $n \times n$ matrices. Let $S = \{A \in \mathbb{R}^{n\times n} \mid A = A^\top\}$ be the subspace of symmetric $n \times n$ matrices. Define the mapping $T : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ by

$$T(X) = X + X^\top$$

for all $X \in \mathbb{R}^{n\times n}$.

(a) Show that T is a linear operator.

(b) Prove that the range of T is equal to S.

(c) Let $n = 2$. Compute the matrix representation of T with respect to the ordered bases

$$E = F = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right).$$

(d) Let $n = 2$. Compute the kernel of T.

3 $(9 = 3 + 3 + 3$ pts) **Change of basis and dimension theorem**

Let P_n denote the $\mathbb{R}$-vector space consisting of polynomials over $\mathbb{R}$ of degree $\leq n$ in the variable x. Fix a subset $\{\alpha_0, \alpha_1, \ldots, \alpha_n\} \subset \mathbb{R}$ consisting of precisely $n+1$ elements. For any $j \in \{0, \ldots, n\}$ define $p_j \in P_n$ by $p_j(x) = \prod_{k \neq j}^{n} \frac{x - \alpha_k}{\alpha_j - \alpha_k}$ (the product over all $k \in \{0, \ldots, n\}$ different from j). Let β be the basis of P_n over $\mathbb{R}$ consisting of the $n+1$ polynomials $1, x, x^2, \ldots, x^n$ and let ϵ be the standard basis of $\mathbb{R}^{n+1}$.

(a) Show that for $i, j \in \{0, 1, \ldots, n\}$ it holds that $p_j(\alpha_i) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$

(b) Explain why $T\colon P_n \to \mathbb{R}^{n+1}$ given by $T(f) = \begin{pmatrix} f(\alpha_0) \\ \vdots \\ f(\alpha_n) \end{pmatrix}$ is surjective.

(c) Find the matrix $A = {}_{\epsilon}[T]_{\beta}$, and explain why A is invertible. (Hint: consider $T(p_j)$ for $j = 0, \ldots, n$.)

4 $(9 = 3 + 3 + 3$ pts) **Inner product and norm**

Consider the $\mathbb{R}$-inner product space P_4 consisting of polynomials over $\mathbb{R}$ of degree ≤ 4 in the variable x; as inner product on P_4 we take $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\,dt$.

(a) Verify that $||x^4 + x + 1||^2 = ||x^4 + 1||^2 + ||x||^2$.

(b) Prove that $|| \cdot ||_2 \colon P_4 \to \mathbb{R}_{\geq 0}$ defined by $||f||_2 := ||\frac{df}{dx}|| + |f(0)|$ is a norm on P_4.

(c) Show that $\langle\langle f, g \rangle\rangle := \frac{1}{4}(||f + g||_2^2 - ||f - g||_2^2)$ is not an inner product on P_4, by calculating $\langle\langle 1, x \rangle\rangle$ and $\langle\langle 1, 1 \rangle\rangle$ and $\langle\langle 1, x + 1 \rangle\rangle$.

4 pts free